Bruhat-Chevalley order on fixed-point-free involutions

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Abstract

The purpose of this paper is twofold. First is to prove that the Bruhat-Chevalley ordering when restricted to the fixed-point-free involutions forms an EL-shellable poset whose order complex triangulates a ball. Second is to prove that the Deodhar-Srinivasan poset is a proper, graded subposet of the Bruhat-Chevalley poset on fixed-point-free involutions.

Keywords: Perfect matchings, symmetric and skew-symmetric matrices, Bruhat-Chevalley ordering, lexicographic shellability, triangulations of balls and spheres.

MSC-2010: 06A11, 14L30, 05E18

1 Introduction

In this manuscript we are concerned with the interaction between two well known subgroups of the special linear group SL_{2n}, namely a Borel subgroup and a symplectic subgroup. Without loss of generality, we choose the Borel subgroup B to be the group of invertible upper triangular matrices, and define the *symplectic group,* Sp_{2n} as the subgroup of fixed elements

of the involutory automorphism $\theta : \mathrm{SL}_{2n} \to \mathrm{SL}_{2n}$, $\theta(g) = J(g^{-1})^\top J^{-1}$, where J denotes the skew form

$$J = \begin{pmatrix} 0 & id_n \\ -id_n & 0 \end{pmatrix}, \tag{1}$$

and id_n is the $n \times n$ identity matrix.

It is clear that B acts by left-multiplication on the symmetric space $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$. We investigate the covering relations of the poset F_{2n} of inclusion relations among the B-orbit closures. It is known since the works of Beilinson-Bernstein [2] and Vogan [23] that such inclusion posets have importance in the study of discrete series representations of the real forms of semi-simple Lie groups.

To further motivate our discussion and help the reader to place our work appropriately we look at a related situation.

It is well known that the symmetric group of permutation matrices, S_m parametrizes the orbits of the Borel group of upper triangular matrices $B \subset \mathrm{SL}_m$ in the flag variety SL_m/B. For $u \in S_m$, let $\dot{u}$ denote the right coset in SL_m/B represented by u. The classical *Bruhat-Chevalley ordering* is defined by $u \leq_{S_m} v \iff B \cdot \dot{u} \subseteq \overline{B \cdot \dot{v}}$ for $u, v \in S_m$.

A permutation $u \in S_m$ is said to be an *involution*, if $u^2 = id$, or equivalently, its permutation matrix is a symmetric matrix. We denote by I_m the set of all involutions in S_m, and consider it as a subposet of the Bruhat-Chevalley poset $(S_m, \leq_{S_m})$. Let m be an even number, $m = 2n$. An involution $x \in I_{2n}$ is called *fixed-point-free*, if the matrix of x has no non-zero diagonal entries. In [[20], Example 10.4], Richardson and Springer show that there exists a poset isomorphism between F_{2n} and a subposet of fixed-point-free involutions in I_{2n}. Unfortunately, F_{2n} does not form an interval in I_{2n}, hence it does not immediately inherit nice properties therein. In fact, this is easily seen for $n = 2$ from the Hasse diagram of I_4 in Figure 1, in which the fixed point free involutions are boxed.

Let $\leq$ denote the restriction of the Bruhat-Chevalley ordering on F_{2n}. Our first main result is that $(F_{2n}, \leq)$ is "EL-shellable," which is a property that well known to be true for many other related posets. See [6],[7],[11],[15], and [16].

Recall that a finite graded poset P with a maximum and a minimum element is called *EL-shellable*, if there exists a map $f = f_\Gamma : C(P) \to \Gamma$ between the set of covering relations $C(P)$ of P into a totally ordered set Γ satisfying

1. in every interval $[x, y] \subseteq P$ of length $k > 0$ there exists a unique saturated chain

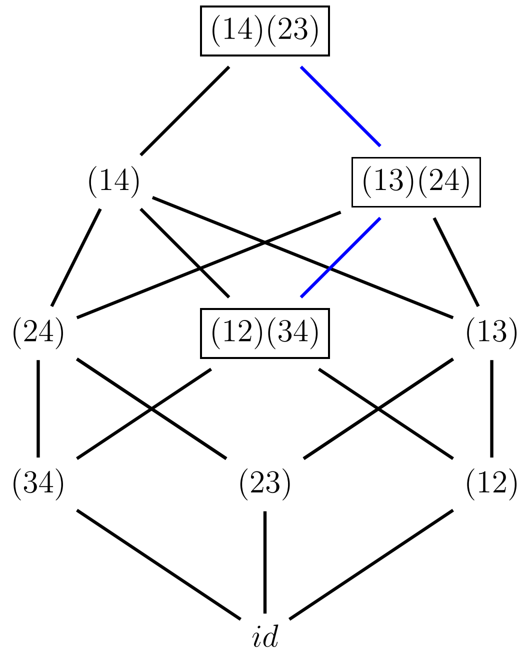

Figure 1: F_4 in I_4

$\mathfrak{c}:\ x_0 = x < x_1 < \cdots < x_{k-1} < x_k = y$ such that the entries of the sequence

$$f(\mathfrak{c}) = (f(x_0, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k)) \tag{2}$$

is weakly increasing.

2. The sequence $f(\mathfrak{c})$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among all sequences of the form $(f(x_0, x_1'), f(x_1', x_2'), \ldots, f(x_{k-1}', x_k))$, where $x_0 < x_1' < \cdots < x_{k-1}' < x_k$.

To highlight the importance of this notion, let us mention the following well known fact. The *order complex* of a poset P is the abstract simplicial complex $\Delta(P)$ whose simplicies are the chains in P. For an EL-shellable poset the order complex is shellable, in particular it implies that $\Delta(P)$ is Cohen-Macaulay [3]. These, of course, are among the most desirable properties of a topological space.

As a corollary of our construction of the EL-labeling of F_{2n}, we prove a special case of a conjecture of A. Hultman that the order complex of F_{2n} triangulates a ball of dimension $n^2 - n - 2$. See Conjecture 6.3, [14]. See [13], also.

In literature there are different versions of lexicographic shellability, and up to today it is known that EL-shellability is the strongest among all. A closely related notion with the

same topological consequences as EL-shellability is called "CL-shellability." In fact, it is known that the EL-shellability implies CL-shellability, see [5]. In [4], Björner and Wachs show that all Coxeter groups, as well as all sets of minimal-length coset representatives in Coxeter groups are CL-shellable.

There are various directions that the results of [4] are extended. For semigroups, in [17], Putcha shows that "J-classes in Renner monoids" are CL-shellable. In an another direction, in [18], Rains and Vazirani show that "quasiparabolic" sets in Coxeter groups are CL-shellable. Note that, in particular, the fixed-point-free involutions form a quasiparabolic set in S_{2n}. Therefore, our result is a strengthening of the corresponding result of [18] in this special case.

One of the reasons the shellability of F_{2n} is not considered before is that there is a closely related EL-shellable partial order studied by Deodhar and Srinivasan in [10]. However, it is noted in [14] (without proofs) that this poset is different than Bruhat-Chevalley ordering on F_{2n}. Let us explain.

Obviously, every $x \in F_{2n}$ is expressible as a product of transpositions. Indeed, let $i_1, \dots, i_n$ denote the list of all numbers from $\{1, \dots, 2n\}$ such that $x(i_r) > i_r$ for $r = 1, \dots, n$. Then $x = (i_1, x(i_1))(i_2, x(i_2)) \cdots (i_n, x(i_n))$. Note that disjoint cycles (hence transpositions) commute, therefore, to insist on the uniqueness of the expression, we require that $i_1 < \cdots < i_n$. In this case, by a change of notation, in place of x we write $[i_1, x(i_1)][i_2, x(i_2)] \cdots [i_n, x(i_n)]$. Let $\tilde{F}_{2n}$ denote the set of all such unique ordered expressions, one for each $x \in F_{2n}$.

The partial ordering of [10], which we call the *Deodhar-Srinivasan partial ordering* and denote by $\leq_{DS}$, is defined as the transitive closure of the following relations.

$y = [c_1, d_1] \dots [c_n, d_n] \in \tilde{F}_{2n}$ is said to be greater than $x = [a_1, b_1] \cdots [a_n, b_n] \in \tilde{F}_{2n}$ in $\leq_{DS}$, if there exist $1 \leq i < j \leq n$ such that

1. y is obtained from x by interchanging b_i and a_j, or

2. y is obtained from x by interchanging b_i and b_j.

A careful inspection of the Hasse diagrams of $(F_{2n}, \leq)$ and $(\tilde{F}_{2n}, \leq_{DS})$ reveals that these two posets are "almost" the same but different. Our second main result is that the rank functions of these posets are the same, and furthermore, the latter is a graded subposet of the former.

The organization of our manuscript is as follows. In Section 2 and 3 we introduce the notation and provide the necessary background. In particular, in Section 3 we review the

notions of "partial involutions" and "partial fixed-point-free involutions" to prove some results that are needed for our analysis. In Section 4, we recall the EL-labeling of Incitti and study covering relations of F_{2n}. In Section 5 we prove our first main result, and in Section 6 we prove our second main result. In Section 7 we show that $\Delta(F_{2n})$ triangulates a ball of dimension $n^2 - n - 2$. We conclude our paper in Section 8 with a short discussion of the various equivalent characterizations of the length function of $\ell_{F_{2n}}$.

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2 Preliminaries

2.1 Poset terminology

Let m be a positive integer. We denote the set $\{1, \dots, m\}$ by $[m]$.

In this paper, all posets are assumed to be finite and assumed to have a minimal and a maximal element, denoted by $\hat{0}$ and $\hat{1}$, respectively.

Recall that in a poset P, an element y is said to *cover* another element x, if $x < y$ and if $x \leq z \leq y$ for some $z \in P$, then either $z = x$ or $z = y$. In this case, we write $y \to x$. Given P, we denote by $C(P)$ the set of all covering relations of P.

An (increasing) *chain* in P is a sequence of distinct elements such that $x = x_1 < x_2 < \cdots < x_{n-1} < x_n = y$. A chain in a poset P is called *saturated*, if it is of the form $x = x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow x_n = y$. A saturated chain in an interval $[x, y]$ is called *maximal*, if the end points of the chain are x and y. Recall also that a poset is called *graded* if all maximal chains between any two comparable elements $x \leq y$ have the same length. This amounts to the existence of an integer valued function $\ell_P : P \to \mathbb{N}$ satisfying

1. $\ell_P(\hat{0}) = 0$,

2. $\ell_P(y) = \ell_P(x) + 1$ whenever y covers x in P.

ℓ_P is called the *length function* of P. In this case, the length of the interval $[\hat{0}, \hat{1}] = P$ is called the *length* of the poset P.

The Möbius function of P is defined recursively by the formula

$$\mu([x,x]) = 1$$
$$\mu([x,y]) = -\sum_{x\leq z<y} \mu([x,z])$$

for all $x \leq y$ in P.

Recall that the *order complex* $\Delta(P)$ of a poset P is the abstract simplicial complex whose elements are chains of P. It is well known that $\mu(\hat{0},\hat{1})$ is equal to the "reduced Euler characteristic" $\widetilde{\chi}(\Delta(P))$ of the topological realization of $\Delta(P)$. See Proposition 3.8.6 in [21]. This fact provides a plenty of motivation to compute the Möbius functions, in general.

Let Γ denote a finite totally ordered poset and let g be a Γ-valued function defined on $C(P)$. Then g is called an *R-labeling* for P, if for every interval $[x,y]$ in P, there exists a unique saturated chain $x = x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_{n-1} \leftarrow x_n = y$ such that

$$g(x_1,x_2) \leq g(x_2,x_3) \leq \cdots \leq g(x_{n-1},x_n). \tag{3}$$

Thus, P is EL-shellable, if it has an R-labeling $g : C(P) \to \Gamma$ such that, for each interval $[x,y]$ in P the sequence (3) is lexicographically smallest among all sequences of the form

$$(g(x,x_2'), g(x_2',x_3'), \ldots, g(x_{k-1}',y)),$$

where $x \leftarrow x_2 \leftarrow' \cdots \leftarrow x_{k-1}' \leftarrow y$.

Suppose P is of length $n \in \mathtt{N}$ with the length function $\ell = \ell_P : P \to \mathtt{N}$. For $S \subseteq [n]$, let P_S denote the subset $P_S = \{x \in P : \ \ell(x) \in S\}$. We denote by μ_S the Möbius function of the poset $\hat{P}_S$ that is obtained from P_S by adjoining a smallest and a largest element, if they are missing.

Suppose also that $g : C(P) \to \Gamma$ is an R-labeling for P. In this case, it is well known that $(-1)^{|S|-1}\mu_S(\hat{0}_{\hat{P}_S},\hat{1}_{\hat{P}_S})$ is equal to the number of maximal chains $x_0 = \hat{0} \leftarrow x_1 \leftarrow \cdots \leftarrow x_n = \hat{1}$ in P for which the sequence $(g(x_0,x_1),\cdots,g(x_{n-1},x_n)$ has descent set S, that is, for which $\{i \in [n] : \ g(x_{i-1},x_i) \geq g(x_{i+1},x_i)\} = S$. See Theorem 3.14.2 in [21].

2.2 Involutions and Borel orbits

Let Sym_n denote the affine space of symmetric matrices and let Sym_n^0 denote its closed subset consisting of symmetric matrices with determinant 1. Similarly, let Skew_{2n} denote the affine

space of skew-symmetric matrices, and let Skew^0_{2n} denote its closed subset consisting of elements with determinant 1.

Let X denote any of the spaces Sym^0_n, or Skew^0_{2n}. Then the special linear group of appropriate rank acts on X via

$$g \cdot A = (g^{-1})^\top A g^{-1}. \tag{4}$$

It follows from Cholesky's decompositions that these actions are transitive (see Appendix B of [12]). It is easy to check that the stabilizer of $J \in \mathrm{Skew}^0_{2n}$ (J is as in (1)) is the symplectic subgroup $\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n}$, and the stabilizer of the identity element $id_n \in \mathrm{Sym}^0_n$ is the special orthogonal group $\mathrm{SO}_n := \{g \in \mathrm{SL}_n : \ gg^\top = id_n\}$.

Thefore, the quotients $\mathrm{SL}_{2n}/\mathrm{Sp}_{2n}$ and $\mathrm{SL}_n/\mathrm{SO}_n$ are canonically identified with the spaces Skew^0_{2n} and Sym^0_n, respectively.

Recall that an $n \times n$ *partial permutation matrix* (or, a *rook matrix*) is a 0/1 matrix with at most one 1 in each row and each column. The set of all $n \times n$ rook matrices is denoted by R_n. In [19], Renner shows that R_n parametrizes the $B \times B$-orbits on the monoid of $n \times n$ matrices. It is known that the partial ordering on R_n induced from the containment relations among the $B \times B$-orbit closures is a lexicographically shellable poset (see [6]). On the other hand, for the purposes of this paper, it is more natural for us to look at the inclusion poset of $B^\top \times B$-orbit closures in R_n, which we denote by $(R_n, \leq_{Rook})$.

A symmetric rook matrix is called a *partial involution*. The set of all partial involutions in R_n is denoted by PI_n. It is shown by Szechtzman in [22] that each Borel orbit in Sym_n contains a unique element of PI_n.

A rook matrix is called a *partial fixed-point-free involution*, if it is symmetric and does not have any non-zero entry on its main diagonal. We denote by PF_{2n} the set of all partial fixed-point-free involutions.

In [8], Cherniavsky shows that the Borel orbits in Skew_{2n} are parametrized by those elements $x \in \mathrm{Skew}_{2n}$ such that

1. the entries of x are either 0,1 or -1,
2. any non-zero entry of x that is above the main diagonal is a $+1$,
3. in every row and column of x there exists at most one non-zero entry.

Note that when -1's in x are replaced by $+1$'s, the resulting matrix $\tilde{x}$ is a partial involution with no diagonal entry. In other words, $\tilde{x}$ is a fixed-point-free partial involution. It is easy

to check that this correspondence is a bijection, hence PF_{2n} parametrizes the Borel orbits in Skew_{2n}.

Containment relations among the closures of Borel orbits in Skew_{2n} define a partial ordering on PF_{2n}. We denote its opposite by $\leq_{Skew}$. Similarly, on PI_n we have the opposite of the partial ordering induced from the containment relations among the Borel orbit closures in Sym_n. We denote this opposite partial ordering by $\leq_{Sym}$.

In [15], Incitti, studying the restriction of the partial order $\leq_{Sym}$ on I_n, finds an EL-labeling for I_n. Let us mention that in a recent preprint, Can and Twelbeck, using an extension of Incitti's edge-labeling show that PI_n is EL-shellable. See [7].

3 Posets R_{2n}, PI_{2n}, PF_{2n},I_{2n} and F_{2n}

There is a combinatorial method for deciding when two elements x and y from $(R_n, \leq_{Rook})$ (respectively, from $(PI_n, \leq_{Sym})$, or from$(PF_{2n}, \leq_{Skew})$) are comparable with respect to $\leq_{Rook}$ (respectively, with respect to $\leq_{Sym}$, or $\leq_{Skew}$). To this end, we denote by $Rk(x)$ the matrix whose i,j-th entry is the rank of the upper left $i \times j$ submatrix of x. Hence, $Rk(x)$ is an $n \times n$ matrix with non-negative integer coordinates. We call $Rk(x)$, the *rank-control matrix* of x.

Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two matrices of the same size with real number entries. We write $A \leq B$ if $a_{i,j} \leq b_{i,j}$ for all i and j. Then

$$x \leq_{Rook} y \iff Rk(y) \leq Rk(x). \tag{5}$$

The same criterion holds for the posets $\leq_{Sym}$ and $\leq_{Skew}$.

3.1 Covering relations for PI_{2n} and PF_{2n}

We recall some fundamental facts about the covering relations of $\leq_{Sym}$ and $\leq_{Skew}$. Our references are [1] and [8]. Let $Rk(x) = (r_{i,j})_{i,j=1}^m$ denote the rank-control matrix of an $m \times m$ matrix x. As a notation we set $r_{0,i} = 0$ for $i = 0, \dots, m$ and define

$$\rho_{\leq}(x) = \#\{(i,j) :\ 1 \leq i \leq j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}, \tag{6}$$

$$\rho_{<}(x) = \#\{(i,j) :\ 1 \leq i < j \leq 2n \text{ and } r_{i,j} = r_{i-1,j-1}\}. \tag{7}$$

Then the length function $\ell_{PF_{2n}}$ of the poset PF_{2n} is equal to the restriction of $\rho_{<}$ to PF_{2n}. Furthermore, y covers x if and only if $Rk(y) \lneq Rk(x)$ and $\ell_{PF_{2n}}(y) - \ell_{PF_{2n}}(x) = 1$.

Similarly, $\ell_{PI_{2n}}$ is the restriction of $\rho_{\leq}$ to PI_{2n}, and that y covers x if and only if $Rk(y) \leqslant Rk(x)$ and $\ell_{I_{2n}}(y) - \ell_{I_{2n}}(x) = 1$.

Lemma 1. *The intersection $PF_{2n} \cap I_{2n}$ is equal to F_{2n}, and furthermore, $(F_{2n}, \leq_{Sym})$ and $(F_{2n}, \leq_{Skew})$ are isomorphic.*

Proof. First claim is straightforward. For the second it is enough to observe that the partial orders $\leq_{Skew}$ and $\leq_{Sym}$ are both given by the same rank-control matrix comparison. Therefore, they restrict to give the same poset structure on F_{2n}. $\square$

Whenever it is clear from the context, we write $(F_{2n}, \leq)$ instead of $(F_{2n}, \leq_{Sym})$ or $(F_{2n}, \leq_{Skew})$.

Remark 2. *It is easy to see that the sets PF_{2n} and I_{2n} have the same cardinality. Indeed, let $x \in PF_{2n}$ be a partial fixed-point-free involution with determinant 0. We denote by $\tilde{x}$ the completion of x to an involution in I_{2n} by adding the missing diagonal entries. For example,*

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \tilde{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $\phi : PF_{2n} \to I_{2n}$ by setting

$$\phi(x) = \begin{cases} \tilde{x} & \text{if } x \in PF_{2n} - F_{2n}, \\ x & \text{otherwise.} \end{cases} \tag{8}$$

It is not difficult to check that ϕ is a bijection between PF_{2n} and I_{2n} such that $\phi(x) = x$ for all $x \in F_{2n}$. However, as pointed out in [8] the posets $(PF_{2n}, \leq_{Skew})$ and $(I_{2n}, \leq_{Sym})$ are not isomorphic. (Compare the Hasse diagram of PF_4 as depicted in Example 5.1 in [8] and the Hasse diagram of the opposite of I_4 as depicted in Figure 2 of [15].)

Lemma 3. *Let $w_0 \in PI_{2n}$ denote the "longest permutation," namely, the $2n \times 2n$ anti-diagonal permutation matrix, and let $j_{2n} \in F_{2n}$ denote the $2n \times 2n$ fixed-point-free involution having non-zero entries at the positions $(1, 2), (2, 1), (3, 4), (4, 3), \ldots, (2n-1, 2n), (2n, 2n-1)$, only. In other words, j_{2n} is the fixed-point-free involution with the only non-zero entries along its super-diagonals. Then*

1. *I_{2n} is an interval in PI_{2n} with the smallest element id_{2n} and the largest element w_0.*

2. *F_{2n} is an interval in PF_{2n} with the smallest element j_{2n} and the largest element w_0.*

Proof. Let $x \in I_{2n}$. Then $(2n, 2n)$-th entry of $Rk(x)$ is equal to $2n$ because x is invertible. On the other hand, if $x \in PI_{2n}$ is an element not contained in I_{2n}, then its rank is less than $2n$. In other words, its $(2n, 2n)$-th entry cannot be $2n$, and therefore, it cannot be greater than or equal to any element of I_{2n}. It follows from (5) that w_0 is the smallest, id_{2n} is the largest element of I_{2n}. Therefore, I_{2n} is an interval.

To prove the second claim, it is enough to prove that j_{2n} is the largest element of F_{2n} because we already know that $F_{2n} = I_{2n} \cap PF_{2n}$. Now, the maximality of j_{2n} follows from induction by using the combinatorial criterion (5).

□

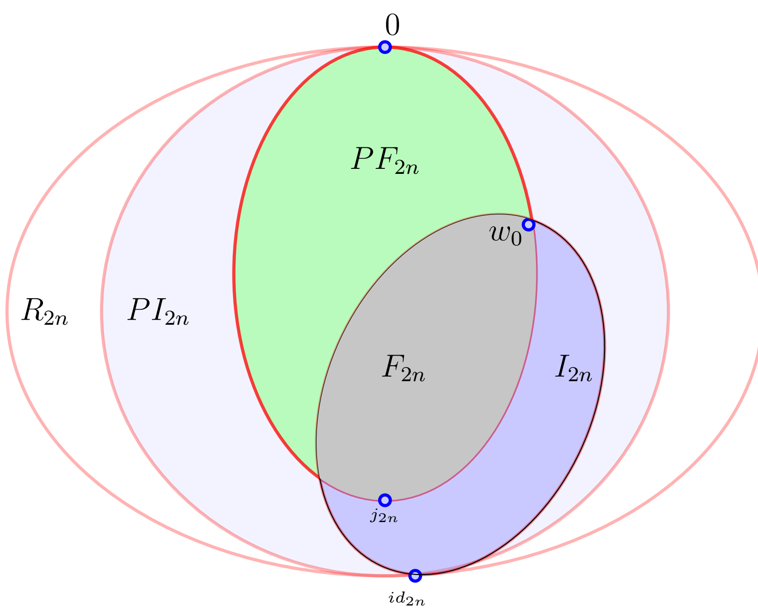


Figure 2: The schematic diagrams of $R_{2n}, PI_{2n}, PF_{2n}, I_{2n}$ and F_{2n}.

3.2 Saturated chains of fixed-point-free involutions

Lemma 4. *Let $x = (x_{ij})_{i,j=1}^{2n}$ be an element in F_{2n}. Then the number of equalities along the main diagonal of $Rk(x)$ is equal to n. In other words, $\rho_{\leq}(x) - \rho_{<}(x) = n$.*

Proof. Since all diagonal entries of x are zero, as we move down along the main diagonal of $Rk(x)$, in each step there are exactly two possibilities: (1) $r_{i+1,i+1} = r_{ii}$, or (2) $r_{i+1,i+1} = r_{ii}+2$. Indeed, as we move from the (i,i)-th position to the $(i+1,i+1)$-th entry, the new minor gains either two new non-zero entries, say $x_{i+1,k} = 1$ and $x_{k,i+1} = 1$, or $x_{i+1,k} = x_{k,i+1} = 0$ for all $1 \leqslant k \leqslant i+1$. The matrix x is invertible, so $r_{2n,2n} = 2n$. Since each step can increase the value r_{ii} only by 2, or does not increase it at all, it has to increase exactly n times and it has to stay the same exactly n times.

□

Proposition 5. *Consider F_{2n} as a subposet of I_{2n} and let $x, y \in F_{2n}$ be two elements such that $x \leq y$. Then there exits a saturated chain in I_{2n} from x to y consisting of fixed-point-free involutions only.*

Proof. First observe that there exists a saturated chain from x to y in PF_{2n} and since F_{2n} is an interval in PF_{2n} this chain consists of fixed-point-free involutions only. For two fixed-point-free involutions u and w of this chain such that w covers u in PF_{2n} we have $Rk(u) < Rk(w)$ and $\ell_{PF_{2n}}(w) - \ell_{PF_{2n}}(u) = 1$. Since $Rk(u) < Rk(w)$, we have $u < w$ in I_{2n} also. By Lemma 4 we have

$$\ell_{PI_{2n}}(w) - \ell_{PI_{2n}}(u) = \ell_{PF_{2n}}(w) + n - (\ell_{PF_{2n}}(u) + n) = 1$$

Therefore w covers u also in I_{2n}, and hence this chain is saturated in I_{2n}, also.

□

4 EL-labeling

4.1 Incitti's EL-labeling

For a permutation $\sigma \in S_n$, a *rise* of σ is a pair of indices $1 \leq i_1, i_2 \leq n$ such that

$$i_1 < i_2 \text{ and } \sigma(i_1) < \sigma(i_2).$$

A rise (i_1, i_2) is called *free*, if there is no $k \in [n]$ such that

$$i_1 < k < i_2 \text{ and } \sigma(i_1) < \sigma(k) < \sigma(i_2).$$

For $\sigma \in S_n$, define its *fixed point set, its exceedance set* and its *defect set* to be

$$I_f(\sigma) = Fix(\sigma) = \{i \in [n] : \sigma(i) = i\},$$
$$I_e(\sigma) = Exc(\sigma) = \{i \in [n] : \sigma(i) > i\},$$
$$I_d(\sigma) = Def(\sigma) = \{i \in [n] : \sigma(i) < i\},$$

respectively.

Given a rise (i_1, i_2) of σ, its *type* is defined to be the pair (a, b), if $i_1 \in I_a(\sigma)$ and $i_2 \in I_b(\sigma)$, for some $a, b \in \{f, e, d\}$. We call a rise of type (a, b) an *ab-rise*. On the other hand, two kinds of *ee*-rises have to be distinguished from each other; an *ee*-rise is called *crossing*, if $i_1 < \sigma(i_1) < i_2 < \sigma(i_2)$, and it is called *non-crossing*, if $i_1 < i_2 < \sigma(i_1) < \sigma(i_2)$.

The rise (i_1, i_2) of an involution $\sigma \in I_n$ is called *suitable* if it is free and if its type is one of the following: $(f, f), (f, e), (e, f), (e, e), (e, d)$.

A *covering transformation*, denoted $ct_{(i_1,i_2)}(\sigma)$, of a suitable rise (i_1, i_2) of σ is the involution obtained from σ by moving the 1's from the black dots to the white dots as depicted in Table 1 of [15].

It is shown in [15] that if τ and σ are two involutions in I_n, then

$$\tau \text{ covers } \sigma \text{ in } \leq_{Sym} \iff \tau = ct_{(i_1,i_2)}(\sigma), \text{ for some suitable rise } (i_1, i_2) \text{ of } \sigma.$$

Let Γ denote the totally ordered set $[n] \times [n]$ with respect to lexicographic ordering. In the same paper, Incitti shows that the labeling defined by

$$f_\Gamma((\sigma, ct_{(i_1,i_2)}(\sigma))) := (i_1, i_2) \in \Gamma \tag{9}$$

is an EL-labeling, hence, $(I_n, \leq_{Sym})$ is a lexicographically shellable poset.

4.2 Covering transformations in F_{2n}

Recall that F_{2n} is a connected graded subposet of I_{2n}. Therefore, its covering relations are among the covering relations of I_{2n}. On the other hand, within F_{2n} we use two types of covering transformations, only. For convenience of the reader, we depict these moves in Figure 3 and Figure 4. These moves correspond to the items numbered 4 and 6 in Table 1 of [15].

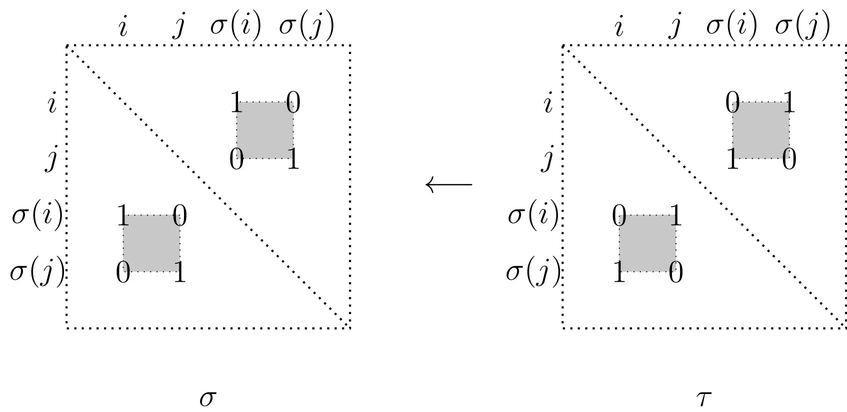


Figure 3: (non-crossing) ee-rise for the covering $\tau \to \sigma$.

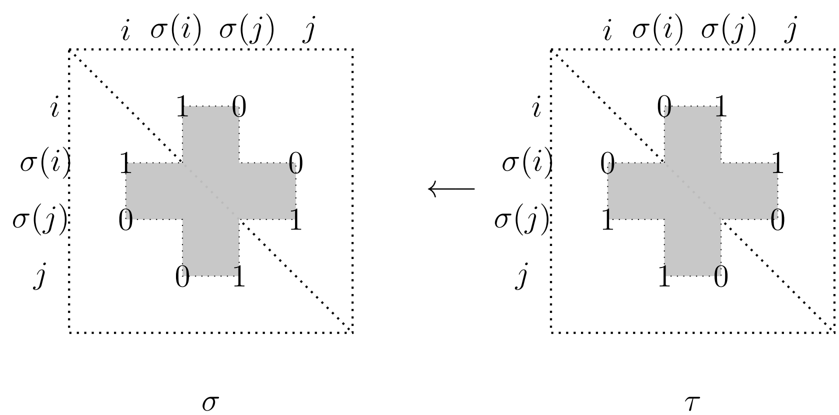


Figure 4: ed-rise for the covering $\tau \to \sigma$.

5 Main Theorem

Theorem 6. *F_{2n} is an EL-shellable poset.*

Proof. Let x and y be two fixed-point-free involutions. By Proposition 5 we know that there exists a saturated chain between x and y that is entirely contained in F_{2n}. Since lexicographic ordering is a total order on maximal chains, there exists a unique largest such chain. We denote it by

$$\mathfrak{c} : x = x_1 < x_2 < \cdots < x_s = y.$$

The idea of the proof is showing that $\mathfrak{c}$ is the unique decreasing chain and therefore by switching the order of our totally ordered set $\mathbb{Z}^2$ obtaining the lexicographically smallest chain which is the unique increasing chain. See Figure 5 for an illustration.

Towards a contradiction assume that $\mathfrak{c}$ is not decreasing. Then, there exist three consecutive terms

$$\sigma = x_{t-1} < \tau = x_t < \gamma = x_{t+1}$$

in $\mathfrak{c}$, such that $f((\sigma,\tau)) < f((\tau,\gamma))$. We have 4 cases to consider.

Case 1: $type(\sigma,\tau) = ee$, and $type(\tau,\gamma) = ee$.
Case 2: $type(\sigma,\tau) = ed$, and $type(\tau,\gamma) = ed$.
Case 3: $type(\sigma,\tau) = ee$, and $type(\tau,\gamma) = ed$.
Case 4: $type(\sigma,\tau) = ed$, and $type(\tau,\gamma) = ee$.

In each of these 4 cases, we either produce an immediate contradiction by showing that either the two moves are interchangeable (hence $\mathfrak{c}$ is not the largest chain), or we construct an element $z \in [x,y] \cap F_{2n}$ which covers σ, and such that $f((\sigma,z)) > f((\sigma,\tau))$. Since we assume that $f(\mathfrak{c})$ is the lexicographically largest Jordan-Hölder sequence, the existence of z is a contradiction, too.

To this end, suppose that the label of the first move is (i,j), and the second move is labeled by (k,l).

Case 1:

We begin with $\{i,j,\sigma(i),\sigma(j)\} \cap \{k,l\} = \emptyset$.

Assume for the moment that $k > j$. Then the covering transformations (k,l) and (i,j) are independent of each other. Therefore, we assume that $i < k < j$. There are four cases;

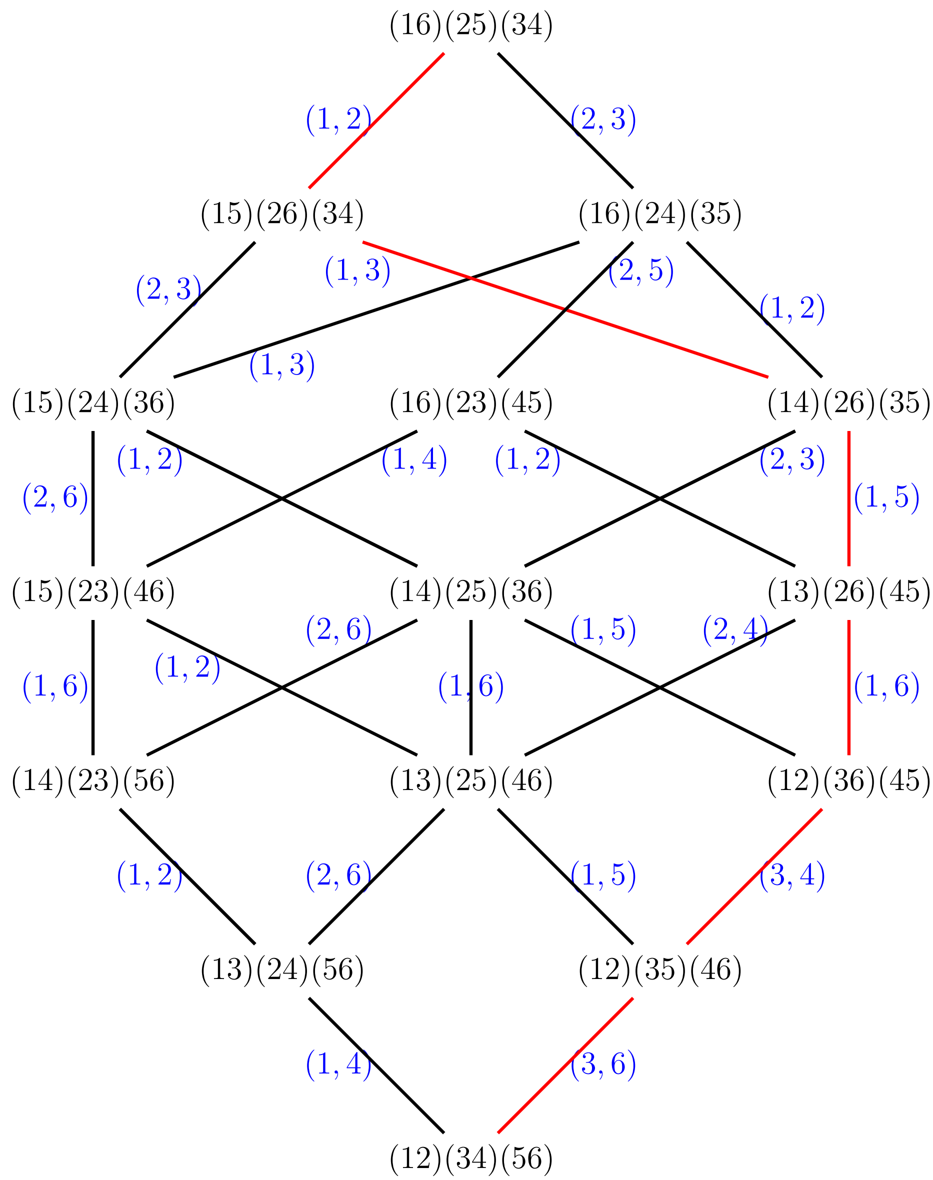


Figure 5: Bruhat-Chevalley order on $\mathrm{SL}_6/\mathrm{Sp}_6$.

$\sigma(k) < j$, $j < \sigma(k) < \sigma(i)$, $\sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(k) > \sigma(j)$. In the first case $(k, \sigma(j))$ is an *ed*-rise for σ with a label bigger than (i, j). This is a contradiction. Similarly, in the second case, (k, j) is an *ee*-rise for σ with a bigger label than (i, j). The third case leads to a contradiction, because in that case (i, j) is not a suitable rise in σ. Finally, in the fourth case the two covering relations (k, l) and (i, j) are independent of each other.

Next we assume that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} \neq \emptyset$.

We observe that if $k = \sigma(i)$, then we have $\tau(k) = \sigma \cdot (i, j) \cdot (\sigma(i), \sigma(j))(k) = j$. Then, we obtain $j < \sigma(i) = k < \tau(k) = j$, which is absurd. Similarly, if $k = \sigma(j)$, then we have $\tau(k) = i$, and from $i < \sigma(j) = k < \tau(k) = i$ we obtain another contradiction.

Next observe that if $l = \sigma(i)$, then we have $\tau(l) = \sigma \cdot (i, j) \cdot (\sigma(i), \sigma(j))(l) = j$, and from $j < \sigma(i) < \sigma(j) = l < \tau(l) = j$ we obtain a contradiction. Likewise, $l = \sigma(j)$ is impossible.

If $i = k$, then, of course we must have $j < l$. In this case we must also have that $\tau(k) = \sigma(j)$. In this case, it is easy to check that $\tau(l) = \sigma(l)$, therefore, (j, l) is an *ee*-rise for σ which is bigger than (i, j), a contradiction. If $j = k$, then we have $\tau(k) = \sigma(i)$. Just as in the previous case, (i, l) is an *ee*-rise for σ. Furthermore, $(i, l) > (i, j)$ gives the contradiction. Finally, if $j = l$, then it is easy to check that (k, j) is an *ee*-rise for σ, therefore, we have another contradiction, and this finishes the proof of the first case.

Case 2:

We begin with the assumption that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} = \emptyset$.

Then $k > i$. If $k > \sigma(j)$, then observe that $l > \tau(l) = \sigma(l) > \tau(k) = \sigma(k) > k > \sigma(j)$. It follows that (k, l) is an *ed*-rise for σ with a bigger label than (i, j), a contradiction.

We proceed with the assumption that $i < k < \sigma(j)$. If $\sigma(k) > j$, then the two moves are interchangeable. If $\sigma(k)$ is in between i and $\sigma(j)$, then $(k, \sigma(j))$ is an *ee*-rise for σ.

We proceed with the assumption that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$, then we have $j < l$. Since τ is obtained from σ by applying the covering transformation (i, j), in this case we see that $\tau(k) = \sigma(j)$. Note also that $\tau(l) = \sigma(l)$. Therefore, $\sigma(j) < \sigma(l) < l$. If $\sigma(l) < j$, then $(\sigma(j), \sigma(l))$ is an *ee*-rise for σ with a label bigger than (i, j), which is a contradiction. Otherwise, $(\sigma(j), l)$ is an *ed*-rise for σ with a label bigger than (i, j), which is another contradiction.

If $k = j$, then since τ is obtained from σ by the covering transformation of (i, j), $\tau(j) = \sigma(i)$. But this is impossible, because (k, l) is an *ed*-rise for τ, and hence $k < \tau(k)$ which

implies that $j = k < \sigma(i)$.

If $k = \sigma(i)$, then $\tau(k) = j$ hence $\sigma(j) < j < \tau(l) = \sigma(l)$. Therefore, $(\sigma(j), l)$ is an *ed*-rise in σ.

If $k = \sigma(j)$, then $k < \tau(k) = i$, which is absurd.

If $l = j$, then we see that (k, l) is an *ed*-rise for σ, which is a contradiction.

If $l = \sigma(i)$, then $l > \tau(l) = j > \sigma(i) = l$, which is absurd.

Finally, if $l = \sigma(j)$, then $\tau(k) = \sigma(k)$ and furthermore $\sigma(k) = \tau(k) < \tau(l) = i$. Therefore, $(k, \sigma(i))$ is an *ed*-rise for σ, which is a contradiction.

Case 3:

We begin with the assumption that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} = \emptyset$, hence $k > i$. If $k > j$, then the order of the covering transformations are interchangeable leading to a contradiction. Therefore we assume that $i < k < j$. If $\sigma(k) > \sigma(j)$, then once again in this case the two moves are interchangeable. On the other hand, if $\sigma(i) < \sigma(k) < \sigma(j)$, then (i, j) is not a suitable rise for σ, which is a contradiction.

If $\sigma(k) < \sigma(i)$, then we consider two cases; $\sigma(k) > j$ and $\sigma(k) < j$. In the former case, either the two moves are interchangeable, or (k, j) is an *ee*-rise for σ with a bigger label than (i, j), hence a contradiction.

In the latter case, we have $i < \sigma(k) < j$. In this case, if $l < \sigma(i)$, then the two moves are interchangeable. If $\sigma(i) < l < \sigma(j)$, then either $\sigma(l)$ is in between i and j or $\sigma(l)$ is greater than j. In the former case, (i, j) is not a suitable rise. If $\sigma(l) > j$, then (k, l) is not a suitable rise for τ, because in this case $\sigma(k) < \tau(j) = \sigma(i) < \sigma(l)$. Now, if $\sigma(j) < l$, then we have two possibilities again; either $\sigma(l) > j$ or $\sigma(l) < j$. In the former case, (k, l) is not a suitable rise for τ. In the latter case, the two moves are interchangeable.

We proceed with the assumption that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$ then (j, l) is an *ed*-rise for σ. Indeed, in this case, $\tau(l) = \sigma(l)$ and we have the inequalities $j < \sigma(j) = \tau(k) < \tau(l) = \sigma(l) < l$.

If $k = j$ then either $\sigma(l) < \sigma(j)$, or $\sigma(l) > \sigma(j)$. In the former case, we see that (i, l) is an *ed*-rise for σ. In the latter case (j, l) is an *ed*-rise for σ.

If $k = \sigma(i)$, then $k < \tau(k) = \sigma \cdot (i, j) \cdot (\sigma(i), \sigma(j))(k) = j$. Since $j < \sigma(i)$, this is a contradiction. Similarly, if $k = \sigma(j)$, then $k < \tau(k) = \sigma \cdot (i, j) \cdot (\sigma(i), \sigma(j))(k) = i$. Since $i < \sigma(i)$, this is a contradiction, also.

If $l = j$, then we obtain a contradiction to the facts that (k, l) is an *ed*-rise, and (i, j) is an *ee*-rise.

If $l = \sigma(i)$, then $(k, \sigma(j))$ is an *ed*-rise for σ, because $k < \sigma(k) = \tau(k) < \tau(l) = j < \sigma(j)$.

If $l = \sigma(j)$, then $(k, \sigma(i))$ is an *ed*-rise for σ, because $k < \sigma(k) = \tau(k) < \tau(l) = i < \sigma(i)$.

Case 4:

We proceed with the assumption that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} = \emptyset$.

Once again, $k > i$. If $k > \sigma(j)$ then the two moves are interchangeable. Therefore we assume that $i < k < \sigma(j)$.

If $\sigma(k) < \sigma(j)$, then (k, j) is an *ed*-rise for σ.

If $\sigma(j) < \sigma(k) < j$ then $(k, \sigma(j))$ is an *ee*-rise for σ.

If $j < \sigma(k)$, then it is easy to check that the two moves are interchangeable.

We proceed with the case that $\{i, j, \sigma(i), \sigma(j)\} \cap \{k, l\} \neq \emptyset$.

If $k = i$, then $j < l$. Since (k, l) is an *ee*-rise for τ, we see that $l < \tau(k) = \sigma(j)$, hence $j < \sigma(j)$. But (i, j) is an *ee*-rise for σ, hence $j > \sigma(j)$; a contradiction.

If $k = j$, then $i < k < \tau(k) = \sigma \cdot (i, j) \cdot (\sigma(i), \sigma(j))(k) = \sigma(i)$; a contradiction.

If $k = \sigma(i)$ then either $l > \sigma(j)$, which implies that $(\sigma(j), l)$ is an *ee*-rise for σ, or $k < l < \sigma(j)$, which implies that $(i, \sigma(l))$ is an *ed*-rise for σ and because $\sigma(l) = \tau(l)$, the label $(i, \sigma(l))$ is bigger than the label (i, j), hence a contradiction.

If $k = \sigma(j)$, then $i < k < \tau(k) = i$, a contradiction.

If $l = j$, then $i < l < \tau(l) = i$, a contradiction.

Similarly, the case $l = i$ is impossible.

If $l = \sigma(i)$, then we have either $\sigma(k) < \sigma(j)$, or $\sigma(j) < \sigma(k)$.

In the first case, if $\sigma(i) < \sigma(k) < \sigma(j)$, then it is easy to check that $i < k < j$, hence (i, j) is not a suitable rise. On the other hand, if $\sigma(k) < \sigma(i)$, we have a contradiction to $\sigma(i) = l < \tau(k) = \sigma(k)$. We proceed with the case $\sigma(k) > \sigma(j)$, then $(k, \sigma(j))$ is an *ee*-rise for σ.

If $l = \sigma(j)$, then $\tau(l) = i$ and $i < \sigma(j)$ which is a contradiction.

Our next step is to prove that no other chain is lexicographically decreasing. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that for any interval of length k there exists a unique decreasing

maximal chain.

Let $[x, y] \subseteq F_{2n}$ be an interval of length $k+1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

be the maximal chain such that $f(\mathfrak{c})$ is the lexicographically largest Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another decreasing chain

$$\mathfrak{c}' : x = x_0 < x'_1 < \cdots < x'_k < x_{k+1} = y.$$

Since the length of the chain

$$x'_1 < \cdots < x'_k < x_{k+1} = y$$

is k, by the induction hypotheses, it is the lexicographically largest chain between x'_1 and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $type(x_0, x_1) = ee$, and $type(x_0, x'_1) = ee$,
Case 2: $type(x_0, x_1) = ed$, and $type(x_0, x'_1) = ed$,
Case 3: $type(x_0, x_1) = ee$, and $type(x_0, x'_1) = ed$,
Case 4: $type(x_0, x_1) = ed$, and $type(x_0, x'_1) = ee$,

In each of these cases we construct a fixed-point-free involution z such that z covers x'_1 and $f((x'_1, z)) > f((x'_1, x'_2))$; contradiction to the induction hypothesis. To this end, let $f((x_0, x_1)) = (i, j)$, $f((x_0, x'_1)) = (k, l)$ and assume that $(k, l) < (i, j)$.

Case 1:

If $k = i$ and hence $l < j$, then (l, j) is an ee-rise for x'_1. Obviously this label is greater than (k, l) hence it is greater than the label of $x'_1 \leftarrow x'_2$, a contradiction.

Let $k < i, l \leq i$. Then (i, j) is a suitable rise for x'_1, which is greater than (k, l). This is a contradiction to the maximality of the chain $\mathfrak{c}'$.

Let $k < i$ and $i < l < j$. If either $x_0(l) < x_0(i)$, or $x_0(k) > x_0(j)$ hold, then (i, j) is a suitable rise for x'_1. If $x_0(i) < x_0(l) < x_0(j)$, then (i, j) is not a free rise, which is absurd. If $x_0(i) < x_0(k) < x_0(j)$, then (i, l) is a suitable rise in x'_1 with $(i, l) > (k, l)$. If $x_0(k) < x_0(i)$, then (k, l) is not a free rise in x_0.

Let $k < i$ and $l > j$. If $x_0(l) < x_0(i)$, then (i, j) a suitable rise for x'_1. If $x_0(i) < x_0(l) < x_0(j)$ then (i, l) is a suitable rise for x_0 with $(i, l) > (i, j)$. If $x_0(l) > x_0(j)$ then (j, l) is a suitable rise for x_0 with $(j, l) > (i, j)$. This contradicts with our assumption that (i, l) is the lexicographically largest covering label for x_0.

Case 2:

Suppose that $k = i$. Then we have $l < j$. If $x_0(l) < x_0(j)$, then (i, j) is not a suitable rise, a contradiction. On the other hand, if $x_0(l) > x_0(j)$, then the *ee*-rise $(x_0(i), x_0(j))$ is an *ee*-rise for x'_1, another contradiction.

Next, suppose that $k < i$. If $x_0(k) > j$, then (i, j) and (k, l) are interchangeable rises.

Therefore, we look at the case when $x_0(j) < x_0(k) < j$. In this case, if $x_0(l) < j$, then (k, l) is not a free rise. On the other hand, if $x_0(l) > j$ then (i, j) and (k, l) are interchangeable rises.

If $x_0(i) < x_0(k) < x_0(j)$, then because (i, j) and (k, l) are free rises for x_0, we must have either $l > j, x_0(l) < x_0(j)$, or $l < j, x_0(l) < x_0(j)$. In both of these cases, (i, j) gives a suitable rise for x'_1 with a larger label than (k, l).

If $i < x_0(k) < x_0(i)$ then (i, j) is interchangeable with (k, l) in x'_1.

If $x_0(k) < i$, then either $x_0(l) < i$ or $l < x_0(i)$. In both cases (i, j) and (k, l) are interchangeable rises for x'_1, which leads to the contradiction that we seek.

Case 3:

In this case, $k = i$ with $l < j$ is not possible because then (k, l) is not a free rise in x_0.

If $k < i$ and $x_0(k) > x_0(j)$, then (i, j) and (k, l) are interchangeable for x'_1.

If $k < i$ and $x_0(i) < x_0(k) < x_0(j)$, then for (k, l) to be a free rise in x_0 we have to have that $x_0(k) < x_0(l) < x_0(j)$. In this case, (i, j) is a suitable rise for x'_1.

Suppose now that $k < i$ and $x_0(k) < x_0(i)$.

If $l = x_0(j)$, then $i < x_0(k) < j$ and therefore, $(i, x_0(j))$ is a suitable rise for x'_1 with a label greater than (k, l).

If $l = x_0(i)$, then $x_0(k) < i$. In this case, let m be the largest integer less than i such that $x_0(i) \leq x'_1(m) < x_0(j)$. (Note that such an m exists because $x_0(k)$ satisfies these conditions.) Then (m, j) is a suitable rise for x'_1.

If $x_0(l) < i$ or if $l < x_0(i)$ then (i, j) is a suitable rise for x'_1 and the contradiction is as

before. If $x_0(i) < l < x_0(j)$, since (i, j) is a suitable rise, we have either $x_0(l) < i$ and we are done by the previous case, or $x_0(l) > j$. In the latter case, we have $x_0(l) < x_0(i)$; otherwise (k, l) is not a free rise. Then, $(i, x_0(l))$ is a suitable *ee*-rise for x_0 with a label greater than (i, j).

If $l > x_0(j)$, then we have three cases; $x_0(l) < i, i < x_0(l) < j$, or $x_0(l) > j$. The first case is already taken care of. For the second possibility, because (k, l) is a free rise for x_0, we have to have that $i < x_0(k) < x_0(l)$. Then (i, j) and (k, l) are interchangeable. Finally, if $x_0(l) > j$, then (i, j) and (k, l) are interchangeable.

Case 4:

Let $k = i, l < j$. Since (k, l) is a non-crossing *ed*-rise we have to have $l < x_0(k) = x_0(i)$. Then (l, j) is a suitable *ed*-rise for x_1' with a label greater than (k, l), a contradiction.

Suppose now that $k < i$.

If $x_0(k) > j$, then (i, j) is a suitable rise for x_1' with a greater label than (k, l), a contradiction.

If $x_0(i) < x_0(k) < j$, then we look at where l is located at.

If $l < i$, then (i, j) and (k, l) are interchangeable.

If $x_0(i) < l < x_0(j)$, then $x_0(l) > j$ since otherwise (i, j) is not a free rise for x_0. It follows that $(l, x_0(j))$ is a suitable rise in x_1', a contradiction.

If $l = x_0(j)$ then $(i, x_0(k))$ is possible in x_1'. If $i < l < x_0(j)$, then $x_0(l) > j$ since otherwise (i, j) is not a free rise for x_0. But then $(l, x_0(j))$ is a suitable rise for x_1'; another contradiction.

If $x_0(i) < l < i$ and also if $x_0(l) < x_0(i)$, then (i, j) and (k, l) are interchangeable.

If $x_0(i) < l < i$ and $x_0(i) > x_0(l) > x_0(j)$, then (i, j) is not a suitable rise.

If $x_0(i) < l < i$ and $x_0(j) < x_0(l)$ then either (i, j) and (k, l) are interchangeable, or (l, j) is a suitable rise.

If $x_0(j) < l$, then because (k, l) is a free rise we must have that $x_0(l) < j$.

If $l > x_0(j)$, then once again (i, j) and (k, l) are interchangeable. This finishes the checking of the cases for $x_0(i) < x_0(k) < j$.

Our final case is when $x_0(k) < x_0(i)$. In this case, we look at where l is located at.

If $l = i$, then let m denote the largest integer $< i$ such that $x_0(i) \leq x_1'(m) < x_0(j)$. Such an m exists because k satisfies these conditions. Then (m, j) is a suitable *ed*-rise in x_1' with

a label greater than (k,l). If $l \neq i$, then we have either $l < i$, or $x_0(l) < x_0(i)$. In both of these cases (i,j) and (k,l) are interchangeable, hence a contradiction. This finishes our proof of the uniqueness, and hence the theorem follows.

□

6 Deodhar-Srinivasan poset vs. $(F_{2n}, \leq)$

As it is mentioned in the introduction, the posets $(\tilde{F}_{2n}, \leq_{DS})$ and $(F_{2n}, \leq)$ are different. Indeed, for $2n = 6$ the Hasse diagrams of these two posets differ by an edge.

In this section we show that $\tilde{F}_{2n}$ is a subposet of F_{2n}. We proceed by recalling the definition of the length function of $\tilde{F}_{2n}$ as defined in [10].

Let $[i_1, j_1] \cdots [i_n, j_n]$ be an element from $\tilde{F}_{2n}$, and let $x \in F_{2n}$ denote the corresponding fixed-point-free involution. The arc-diagram of $x \in F_{2n}$ is defined as follows. We place the numbers 1 to $2n$ on a horizontal line. We connect the numbers i and j by a concave-down arc, if $j = x(i)$. Let $c(x)$ denote the number of intersection points of all arcs.

The length function $\ell_{\tilde{F}_{2n}}$ of $\tilde{F}_{2n}$ is given by

$$\ell_{\tilde{F}_{2n}}([i_1, j_1] \cdots [i_n, j_n]) = \sum_{t=1}^{n} (j_t - i_t - 1) - c(\pi)\,.$$

See Theorem 1.3 in [10].

Our first observation is that $\ell_{\tilde{F}_{2n}}$ is in fact an inversion number. To this end, for x as above, let us define the *modified inversion number* of x to be the number of inversions in the word $i_1 j_1 i_2 j_2 \cdots i_n j_n$, and denote it by $\widetilde{inv}(x)$. Note that i_1 is always 1 for fixed point free involutions.

Proposition 7. *Let $[i_1, j_1] \cdots [i_n, j_n] \in \tilde{F}_{2n}$, and let $x \in F_{2n}$ be the corresponding fixed-point-free involution. Then*

$$\widetilde{inv}(x) = \ell_{\tilde{F}_{2n}}([i_1, j_1] \cdots [i_n, j_n]).$$

Proof. An inversion in the word $i_1 j_1 i_2 j_2 \cdots i_n j_n$ is either the pair (j_p, i_q), or the pair (j_p, j_q), where $p < q$ and $j_p > i_q$, or $j_p > j_q$, respectively.

We count inversions in another way. If (i_t, j_t) is a transposition that appear in $[i_1, j_1] \cdots [i_n, j_n]$ of x, then $j_t - i_t - 1 = \#\{m \,:\, m \in \mathbb{N}\,,\, i_t < m < j_t\}$. On the other hand, each number $m \in \{i_t + 1, \ldots, j_t - 1\}$ appears as an entry in another transposition of $[i_1, j_1] \cdots [i_n, j_n]$.

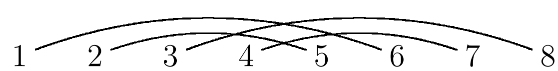
There are three possible cases:

1. the number m is involved in the transposition (a, m), where $a < i_t < m$;
2. the number m is involved in the transposition (a, m), where $i_t < a < m$;
3. the number m is involved in the transposition (m, b), where $m < b$.

In the first case the pair (j_t, m) is not an inversion. Notice that when $a < i_t$, the arc corresponding to the transposition (a, m) crosses the arc corresponding to the transposition (i_t, j_t). In cases 2 and 3, we have the inversion pair (j_t, m) always. For Case 3, whether b is greater than j_t or not is not important. So, to get the number of inversion pairs $(j_t, *)$ we have to subtract from $j_t - i_t - 1$ the number of intersections of the arc (i_t, j_t) with the arcs (a, m), where $a < i_t < m < j_t$. Counting the inversions by summing up the contributions of all the transpositions (i_t, j_t) proves our statement. □

Let us illustrate our proof by an example.

Example 8. *Take* $x = (1, 6)(2, 5)(3, 8)(4, 7) \in S_8$.



Start with the transposition $(1, 6)$. *The numbers between 1 and 6 are 2,3,4,5. All the pairs* $(6, 2)$, $(6, 3)$, $(6, 4)$, $(6, 5)$ *are inversions of the word* 16253847*: 2,3,4 are involved in transpositions of the form* $(m, *)$ *which is case (3) in our proof and always gives an inversion, 5 is involved in transposition* $(2, 5)$, *it is case (2) since* $1 < 2$, *so it also gives an inversion. Now take the transposition* $(2, 5)$. *Both of the numbers 3,4 which are between 2 and 5 are involved in transpositions of case (3),* $(3, 8)$ *and* $(4, 7)$ *and so both of them give inversions* $(5, 3)$ *and* $(5, 4)$. *Now consider the transposition* $(3, 8)$. *The pair* $(8, 4)$ *is an inversion, it is case (3) since 4 is involved in the transposition* $(4, 7)$. *The pair* $(8, 7)$ *also is an inversion since 7 is involved in the transposition* $(4, 7)$ *and* $3 < 4$, *which belongs to case (2). But the pairs* $(8, 5)$ *and* $(8, 6)$ *are not inversions since 5 and 6 are involved in transpositions* $(2, 5)$ *and* $(1, 6)$, *where* $1 < 3$ *and* $2 < 3$ *and so both of them are of case (1). By the same reason when we consider the last transposition of* x *which is* $(4, 7)$, *the pairs* $(7, 5)$ *and* $(7, 6)$ *are not inversions, they belong to case (1). So, summing up, we have four inversions of the form* $(6, *)$ *contributed by the transposition* $(1, 6)$, *two inversions of the form* $(5, *)$ *contributed by the transposition* $(2, 5)$ *and two inversions of the form* $(8, *)$ *contributed by the transposition*

$(3,8)$. *Thus,* $\widetilde{inv}(x) = 4+2+2 = 8$. *From the arc diagram depicted above we see that* $c(x) = 4$. *Hence,*

$$\ell_{\tilde{F}_{2n}}(x) = (6-1-1)+(5-2-1)+(8-3-1)+(7-4-1)-4 = 4+2+4+2-4 = 8.$$

So, we see that $\widetilde{inv}(x) = \ell_{\tilde{F}_{2n}}(x)$ *as it is expected.*

Corollary 9. *The length functions of* $(F_{2n}, \leq)$ *and* $(\tilde{F}_{2n}, \leq_{DS})$ *are the same.*

Proof. This follows from Proposition 7 above combined with Proposition 6.2 of [8]. $\square$

Recall that $y \to x = [a_1, b_1] \cdots [a_n, b_n]$ in $\tilde{F}_{2n}$, if $\ell_{\tilde{F}_{2n}}(y) = \ell_{\tilde{F}_{2n}}(x) + 1$ and there exists $1 \leq i < j \leq n$ such that

1. y is obtained from x by interchanging b_i and a_j, where $b_i < a_j$, or
2. y is obtained from x by interchanging b_i and b_j,where $b_i < b_j$.

We call these interchanges *type 1* and *type 2*, respectively. Note that, in a type 1 covering relation we have the inequalities $a_i < b_i < a_j < b_j$. The inequalities of type 2 are $a_i < a_j < b_i < b_j$.

Note also that an arbitrary interchange of the entries in x does not always result in another element of $\tilde{F}_{2n}$. This is because of the ordering of the a_i's. For example, as it is seen from Figure 3 of [10], there is no edge between the elements $[1,2][3,6][4,5]$ and $[1,4][2,5][3,6]$. On the other hand, it is easy to check using rank-control matrices that the corresponding involution $x = (1,2)(3,6)(4,5)$ is covered by $y = (1,4)(2,5)(3,6)$.

Theorem 10. *Covering relations of the poset* $\tilde{F}_{2n}$ *are among the covering relations of* F_{2n}.

Proof. It suffices to prove that a type 1 covering relation of $\tilde{F}_{2n}$ corresponds an *ed*-rise, and a type 2 covering relation of $\tilde{F}_{2n}$ corresponds to an *ee*-rise in F_{2n}.

Let $\tilde{x} = [a_1, b_1][a_2, b_2] \ldots [a_n, b_n]$ be an element from $\tilde{F}_{2n}$ and $x \in F_{2n}$ be the corresponding fixed-point-free element. Suppose we have the inequalities $a_i < a_j < b_i < b_j$, and $\tilde{y} \in \tilde{F}_{2n}$ is obtained from $\tilde{x}$ by a type 2 interchange. Then

$$\tilde{y} = [a_1, b_1][a_2, b_2] \ldots [a_i, b_j] \ldots [a_j, b_i] \ldots [a_n, b_n]$$

It is straightforward to check that the corresponding y is obtained from x by moving the non-zero entries at the positions (a_i, b_i) and (a_j, b_j) (as well as the corresponding symmetric entries) to the positions (a_i, b_j) and (a_j, b_i) (and to the positions of the corresponding symmetric entries). This is an *ee*-rise for x.

Similarly, suppose we have $a_i < b_i < a_j < b_j$ and $\tilde{y} \in F_{2n}$ is obtained from $\tilde{x}$ by a type 1 move. In the matrix of x, there are non-zero entries at the positions $(a_i, b_i), (a_j, b_j)$, as well as at the corresponding symmetric positions. Then,

$$\tilde{y} = [a_1, b_1][a_2, b_2] \dots [a_i, a_j] \dots [b_i, b_j] \dots [a_n, b_n],$$

and the corresponding element y has non-zero entries at the positions $(a_i, a_j), (b_i, b_j)$, as well as at their symmetric positions. This is an *ed*-rise for x.

Therefore, the covering relations of $\tilde{F}_{2n}$ are among the covering relations of F_{2n}, hence the proof is finished.

□

7 The order complex of F_{2n}

Recall from Section 2.1 that the Möbius function is equal to the reduced Euler characteristic of the topological realization of the order complex of the poset P, and moreover, this number is found by counting the number of maximal chains in P.

In this section by applying these considerations to $P = F_{2n}$, we prove that the order complex $\Delta(F_{2n})$ is homotopy equivalent to a ball of dimension $n(n-1) - 2$.

Theorem 11. *The order complex $\Delta(F_{2n})$ triangulates a ball of dimension $n(n-1) - 2$.*

Proof. We know from [9] that if in a pure shellable complex Δ each $\dim \Delta - 1$ dimensional face lies in at most two maximal faces, then Δ triangulates a sphere or a ball.

We also know that the Bruhat-Chevalley poset I_{2n} is Eulerian, hence every interval of length 2 has 4 elements. Since F_{2n} is a subposet of I_{2n}, every interval of length 2 in F_{2n} has at most 4 elements. This implies that in the order complex $\Delta(F_{2n})$, each $\dim \Delta(F_{2n}) - 1$ dimensional face lies in at most two maximal faces, hence it triangulates a sphere or a ball of dimension

$$\dim \Delta(F_{2n}) = \ell(F_{2n}) = 2\binom{n}{2} - 2 = n(n-1) - 2.$$

To see that it is a ball, we show that the reduced Euler characteristic of $\Delta(F_{2n})$ is 0. Thus, by the discussion above, it is enough to show that there is no maximal chain $j_{2n} \leftarrow x_1 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow w_0$ such that $f(j_{2n}, x_1) > f(x_2, x_1) > \cdots > f(x_{m-1}, w_0)$, where $f : C(F_{2n}) \to \Gamma$ is the EL-labeling that is constructed implicitly in the proof of Theorem 6. Indeed, f is

obtained from Incitti's EL-labeling $g : C(I_{2n}) \to \Gamma$, by reversing the order of the totally ordered set Γ of pairs (i,j), $1 \leq i,j \leq 2n$ ordered lexicographically. Therefore, it is enough to show that there is no maximal chain

$$\mathfrak{c}: \ j_{2n} \leftarrow x_1 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow w_0 \tag{10}$$

in F_{2n} such that $g(j_{2n}, x_1) < g(x_2, x_1) < \cdots < g(x_{m-1}, w_0)$. On the other hand, since g is an EL-labeling, we see that $\mathfrak{c}$ is the unique such chain whose sequence of labels is lexicographically smallest among all such chains in the interval $[j_{2n}, w_0]$ in I_{2n}. Thus, the proof is finished once we show that $\mathfrak{c}$ does not lie in F_{2n}.

It is easy to verify our claim directly in the case of $n = 2$. For $n \geq 3$, let $C(j_{2n})$ denote the set

$$C(j_{2n}) = \{g(j_{2n}, z) \in \Gamma : \ z \in I_{2n} \text{ and } j_{2n} = (1,2)(3,4)\cdots(2n-1,2n) \leftarrow z\}.$$

Observe that $\min C(j_{2n}) = g(j_{2n}, z) = (1,3)$ with $z = (1,4)(5,6)(7,8)\cdots(2n-1,2n)$. Therefore, x_1 of $\mathfrak{c}$ has to be equal to z. Since z has a fixed point, $\mathfrak{c}$ does lie in F_{2n}.

□

8 Final Remarks

In Section 6 we show that the length functions of F_{2n} and $\tilde{F}_{2n}$ are the same. As a corollary we see that

Corollary 12. *For $m \in \mathbb{N}$, let $[m]_q$ denote its q-analogue $1 + q + \cdots + q^{m-1}$. Then, the length-generating function $\sum_{x \in F_{2n}} q^{\ell_{F_{2n}}(x)}$ of F_{2n} is equal to*

$$[2n-1]_q!! := [2n-1]_q[2n-3]_q \cdots [3]_q[1]_q.$$

Proof. Follows from Corollary 2.2 of [10]. □

We should mention here that the conclusion of the above corollary is obtained by other combinatorial methods by A. Avni in his M.Sc. thesis at Bar-Ilan University.

It turns out there is another simple characterization of $\ell_{F_{2n}}$, which seem to be known to the experts. Although it is not difficult to prove, since we could not locate it in the literature we record its proof in here for the sake of completeness.

Proposition 13. *Let $x \in S_{2n}$ be a fixed point free involution. Then*

$$\widetilde{inv}(x) = \ell_{F_{2n}}(x) = \ell_{\tilde{F}_{2n}}(x) = \frac{inv(x) - n}{2},$$

where $inv(x) = |\{(i,j) : \ i < j \text{ and } x(i) > x(j)\}|$.

Proof. Let $x \in F_{2n}$. The second equality is shown to be true in Section 6. The first equality follows from Proposition 6.2 of [8]. It remains to show the third equality.

In Lemma 4 we show that $\ell_{I_{2n}}(x) - \ell_{F_{2n}}(x) = n$. On one hand, we know that $\ell_{I_{2n}}(x) = \frac{exc(x)+inv(x)}{2}$, where $exc(x) = |\{i : i < x(i)\}|$ (see [15]). On the other hand, if $x \in F_{2n}$, then $exc(x) = n$. Therefore,

$$\ell_{F_{2n}}(x) = \ell_{I_{2n}}(x) - n = \frac{n + inv(x)}{2} - n = \frac{inv(x) - n}{2}.$$

$\square$

Example 14. *Let $x = (1,8)(2,6)(3,5)(4,7) \in S_8$. Counting inversions in the word* 18263547 *we obtain $\widetilde{inv}(x) = 10$. On the other hand, written in one line notation $x = 86573241$. We see that $inv(x) = 24$. We are in S_8, so $n = 4$. Indeed we have $10 = \frac{24-4}{2}$.*

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